

On cosmologically designed modified gravity theories

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Versions of parameterized pseudo-Newtonian gravity theories specially designed for cosmology have been introduced in recent cosmology literature. The modifications demand a zero-pressure fluid in the context of versions of modified Poisson-like equation with two different gravitational potentials. We consider such modifications in the context of relativistic gravity theories where the action is a general algebraic function of the scalar curvature, the scalar field, and the kinetic term of the field. In general it is not possible to isolate the zero-pressure fluid component simultaneously demanding a modification in the Poisson-like equation. Only in the small-scale limit we can realize some special forms of the attempted modifications. We address some loopholes in the possibility of showing non-Einstein gravity nature based on pseudo-Newtonian modifications in the cosmological context. We point out that future observations of gravitational weak lensing together with velocity perturbation can potentially test the validity of Einstein's gravity in cosmology context.

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I. INTRODUCTION

In parallel with recent growth of cosmologically relevant observational data, theoretical cosmology also has been growing rapidly. Theories meet with observations mostly in the linearly perturbed stage of the large-scale cosmological structures. In our present understanding, the simple theoretical model of currently Λ CDM (cosmological constant Λ as the dark energy together with the cold dark matter) dominated model, together with inflation generated initial fluctuations by a single field, has enough parameter space to amply cope with current observations. Recently, however, motivated by potential future precision data, more complicated models for the dark energy sector have been pursued in the literature. These include a fluid model with exotic equation of states, a scalar field with diverse potentials, modified relativistic gravity theories, etc; see Ref. [1] for recent reviews.

One recent attempt concerns modifying perturbation equations by some unknown parameters in the level of modified gravity or pseudo-Newtonian gravity [2–41]. Effects of the parameters on the cosmic microwave background radiation, the weak lensing shear field and the growth of large-scale structures, have been studied, and constraints on the parametrized modified gravity models based on current and future observational data have been given. Proper theoretical justification of such attempts would be desirable. One way of justifying the situation is to show that such modifications are allowed or can be implemented in the relativistic gravity theories based on the action formulation. In this work we will examine $f(R, \phi, X)$ gravity as a potential candidate. The $f(R, \phi, X)$ gravity is presented as an action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} f(R, \phi, X) + L_m \right], \quad (1)$$

where f is a general algebraic function of the scalar curvature R , a scalar field ϕ and the kinetic combination $X \equiv \frac{1}{2} \phi^{,c} \phi_{,c}$; L_m is the matter Lagrangian. This includes $f(R)$ gravity and many other gravity theories like Brans-Dicke theory, scalar-tensor theory, non-minimally coupled scalar field theories, etc, as cases.

We will show that in general the pseudo-Newtonian modifications cannot be accommodated in our considered generalized gravity theories, see Sec. II. Only in the small scales far inside the horizon we often have a closed form second-order differential equation of CDM (or baryon) density perturbation in a certain gauge condition with specific effective gravitational constant, see Sec. III. Such small scale coincidences in several modified gravity theories may have motivated the pseudo-Newtonian approach. We point out some loopholes in the pseudo-Newtonian approach, see Sec. IV. Even in Einstein's gravity theory with a minimally coupled scalar field as a dynamical dark energy, we can easily introduce the field potential where the dynamical nature of dark energy is important in perturbation evolution. We believe it is desirable to explain the observational data based on theoretically motivated models, and also based on proper treatment of the complete set of equations provided in such models. In Sec. IV we also point out future observations which can be used to test potentially Einstein gravity nature.

The Appendix presents the complete set of equations in the $f(R, \phi, X)$ gravity in the presence of additional matter. These will be useful in future numerical study of perturbation evolution in such gravity theories. We set $c \equiv 1 \equiv 8\pi G$.

II. VALIDITY OF COSMOLOGICALLY MODIFIED PSEUDO-NEWTONIAN GRAVITY

Several versions of phenomenologically modified gravity theories specially designed for cosmological purpose have been introduced in the literature [2–5, 7–11, 14–19, 21, 22, 24, 26–29, 31, 33, 34, 38, 38, 39, 41]. The modified gravity *assumes*

$$\varphi_\chi = -\bar{\eta}\alpha_\chi, \quad (2)$$

$$\frac{k^2}{a^2}\alpha_\chi = -\frac{1}{2}\bar{\mu}\delta\mu_v, \quad (3)$$

where $\varphi_\chi \equiv \varphi - H\chi$ and $\alpha_\chi \equiv \alpha - \dot{\chi}$ are gauge-invariant combinations which are the same as φ and α , respectively, in the zero-shear gauge ($\chi \equiv 0$); $\delta\mu_v \equiv \delta\mu - \dot{\mu}av$ is a gauge-invariant combination which is the same as $\delta\mu$ in the comoving gauge ($v \equiv 0$). In Bardeen's notation [42] we have $\varphi_\chi = \Phi_H$, $\alpha_\chi = \Phi_A$, and $\delta\mu_v = \mu\epsilon_m$. The gauge-invariant combinations α_χ and φ_χ can be interpreted as the Newtonian and the post-Newtonian potentials, respectively [43]. The variables $\bar{\eta} \equiv \bar{\eta}(k, t)$ and $\bar{\mu} \equiv \bar{\mu}(k, t)$ are new free parameters introduced to modify Einstein's gravity where $\bar{\eta} \equiv 1 \equiv \bar{\mu}$. Later we will show that we could have $\bar{\mu} \neq 1$ even in Einstein's gravity.

Our metric and energy-momentum tensor conventions are

$$ds^2 = -(1 + 2\alpha)a^2d\eta^2 - 2a^2\beta_{,\alpha}d\eta dx^\alpha + a^2 \left[g_{\alpha\beta}^{(3)}(1 + 2\varphi) + 2\gamma_{,\alpha|\beta} \right] dx^\alpha dx^\beta, \quad (4)$$

with $\chi \equiv a\beta + a^2\dot{\gamma}$, and

$$T_0^0 = -\mu - \delta\mu, \quad T_\alpha^0 = -(\mu + p)\frac{1}{k}v_{,\alpha}, \\ T_\beta^\alpha = (p + \delta p)\delta_\beta^\alpha + \frac{1}{a^2} \left(\Pi^\alpha{}_\beta - \frac{1}{3}\delta_\beta^\alpha \Pi^\gamma{}_\gamma \right). \quad (5)$$

A vertical bar is a covariant derivative based on the comoving three-space metric $g_{\alpha\beta}^{(3)}$ of the Robertson-Walker spacetime. The complete sets of scalar-type perturbation equations in Einstein's gravity and in the $f(R, \phi, X)$ gravity are presented in the Appendix A and B, respectively.

It is important to notice that, in the presence of multiple component of fluids or a fluid in the context of generalized relativistic gravity theories, the fluid quantities appearing in Eqs. (3) and (5) are collective ones, see Eqs. (53) in Einstein's gravity, and Eqs. (61) and (63) in the $f(R, \phi, X)$ gravity. In the following, except in the Appendix A and B, we *assume* a flat background, thus set $K = 0$.

A. Single component situation

We consider a single component in an yet unspecified generalized gravity, and accept [9, 26]

$$\dot{\delta}_\chi = -\frac{k}{a}v_\chi - 3\dot{\varphi}_\chi, \quad (6)$$

$$\dot{v}_\chi + H v_\chi = \frac{k}{a}\alpha_\chi, \quad (7)$$

where $\delta \equiv \delta\mu/\mu$. These equations follow from Eqs. (50) and (45), and Eq. (51), respectively, by *demanding* $p = 0$ and $\delta p = 0 = \Pi$. As we consider a single component, all fluid quantities are about that single fluid supported by the nature of generalized gravity. In this case, because we assume $\Pi = 0$, from Eq. (48), and Eqs. (46) and (47), respectively, we inevitably end up with

$$\varphi_\chi = -\alpha_\chi, \quad (8)$$

$$\frac{k^2}{a^2}\alpha_\chi = -\frac{1}{2}\delta\mu_v. \quad (9)$$

In the $f(R, \phi, X)$ gravity context we have $\mu = \mu_X$, $p = p_X$, $\delta\mu = \delta\mu_X$, $\delta p = \delta p_X$. $(\mu + p)v = (\mu_X + p_X)v_X$, and $\Pi = \Pi_X$; the subindex X indicates the contribution from $f(R, \phi, X)$ gravity sector; the fluid quantities of the X -component are presented in Eqs. (57) and (62). This apparently implies $\bar{\eta} = 1 = \bar{\mu}$. That is, in the context of our generalized relativistic gravity theories the phenomenological modifications in Eqs. (2) and (3) with $\bar{\eta} \neq 1$ or $\bar{\mu} \neq 1$ are *not* allowed.

Even in the case such a fluid with $\bar{\mu} = 1 = \bar{\eta}$ is implemented in the context of $f(R, \phi, X)$ gravity, thus $\mu = \mu_X \propto a^{-3}$ etc, those are at best uninteresting, because the nontrivial solutions in $f(R, \phi, X)$ gravity should not affect the Einstein's gravity nature of the metric and fluid variables.

B. Multi-component situation

Now we consider a CDM component in an yet unspecified generalized gravity theory, and accept

$$\dot{\delta}_{c\chi} = -\frac{k}{a}v_{c\chi} - 3\dot{\varphi}_\chi, \quad (10)$$

$$\dot{v}_{c\chi} + H v_{c\chi} = \frac{k}{a}\alpha_\chi, \quad (11)$$

where a subindex c indicates the CDM component. Equations (10) and (11) follow from Eqs. (54) and (45), and Eq. (55), respectively, by *demanding* $p_c = 0$ and $\delta p_c = 0 = \Pi_c$. In this case, from Eq. (48), and Eqs. (46) and (47), respectively, we have

$$\varphi_\chi = -\alpha_\chi - \Pi, \quad (12)$$

$$\frac{k^2}{a^2}\varphi_\chi = \frac{1}{2}\delta\mu_v. \quad (13)$$

Thus, Eqs. (2) and (3) demand $\Pi = (\bar{\eta} - 1)\alpha_\chi$ and

$$\bar{\mu} = \frac{1}{\bar{\eta}}. \quad (14)$$

This is apparently a stringent relation required between the two factors introduced in Eqs. (2) and (3).

If a CDM component is present in the context of $f(R, \phi, X)$ gravity, from Eqs. (61) and (62) we have $\Pi = \Pi_X = \delta F_\chi / F$; $\delta F_\chi \equiv \delta F - \dot{F}\chi$ is a gauge-invariant combination which is the same as δF in the zero-shear gauge ($\chi \equiv 0$); we have $F \equiv \partial f / \partial R$. Thus, Eq. (14) demands

$$\frac{1}{F}\delta F_\chi = (\bar{\eta} - 1)\alpha_\chi. \quad (15)$$

This is a strong physical condition on the nature of a gauge-invariant combination δF_χ .

More seriously, however, it is important to notice that $\delta\mu_v$ in Eq. (13) is a collective density perturbation in the collective comoving gauge ($v \equiv 0$); i.e., we have

$$\delta\mu_v \equiv \delta\mu + 3\frac{a}{k}H(\mu + p)v, \quad (16)$$

where $\delta\mu$ and $(\mu + p)v$ are sum of the fluid quantities, see Eqs. (53) and (63). On the other hand, the CDM density perturbation in the CDM-comoving gauge is $\delta\mu_{cv_c} \equiv \delta\mu_c + 3(a/k)H\mu_c v_c = \delta\mu_{c\chi} + 3(a/k)H\mu_c v_{c\chi}$. If one of the components is due to generalized gravity theory like $f(R, \phi, X)$ gravity, the combination becomes more complicated. If we consider the CDM in $f(R, \phi, X)$ gravity, from Eq. (61) we have

$$\begin{aligned} \delta\mu &= \frac{1}{F}\delta\mu_c + \delta\mu_X - \frac{\mu_c}{F^2}\delta F, \\ (\mu + p)v &= \frac{1}{F}\mu_c v_c + (\mu_X + p_X)v_X, \\ \delta p &= \delta p_X, \quad \Pi = \Pi_X. \end{aligned} \quad (17)$$

Thus, we have

$$\delta\mu_v = \frac{1}{F}\delta\mu_{cv_c} + \delta\mu_{Xv_X} - \frac{\mu_c}{F^2}\delta F_{v_X}, \quad (18)$$

where

$$\begin{aligned} \delta\mu_{cv_c} &\equiv \delta\mu_c - \dot{\mu}_c \frac{a}{k}v_c, \quad \delta\mu_{Xv_X} \equiv \delta\mu_X - \dot{\mu}_X \frac{a}{k}v_X, \\ \delta F_{v_X} &\equiv \delta F - \dot{F} \frac{a}{k}v_X. \end{aligned} \quad (19)$$

Therefore, unless we have $F = 1$ and $\delta\mu_{Xv_X} - (\mu_c/F^2)\delta F_{v_X} = 0$, both of which are quite nontrivial physical conditions on the nature of the generalized gravity, it is not possible to interpret $\delta\mu_v$ in Eq. (3) as $\delta\mu_{cv_c}$ which is $\delta\mu_c$ in the CDM-comoving gauge ($v_c \equiv 0$). It is not likely that after implementing these conditions we have meaningful nontrivial results. For example, we can show that, in the $f(R)$ gravity case, implementing $F = 1$, $\delta\mu_X - (\mu_c/F^2)\delta F = 0$ and $v_X = 0$ simply leads to an inconsistent equation for the background, thus impossible to implement.

Later, in the small-scale limit, in Eq. (27) we will show

$$\delta\mu_v \simeq \frac{1}{F}\delta\mu_{cv_c} + \frac{\bar{\eta} - 1}{\bar{\eta} + 1} \frac{1}{F}\delta\mu_{cv_c}. \quad (20)$$

Thus, compared with Eq. (18), apparently the contribution from X -component is important in our generalized gravity theory.

In the context of generalized gravity theories, in general, it is unavoidable (except for the cosmological constant case) that we have additionally coupled second-order differential equation for the X -component representing the $f(R, \phi, X)$ gravity sector. The complete sets of equations of CDM perturbation in the context of $f(R, \phi, X)$ gravity, which inevitably lead to a fourth-order differential equation, are presented in the Appendix C. In the zero-shear gauge Eqs. (10) and (11) together with Eqs. (70) and (71) are the complete sets we need. In the Appendix C we also present the complete sets of equations in the uniform-field gauge and the CDM-comoving gauge. In all cases we end up having fourth-order differential equations.

III. SMALL-SCALE LIMIT APPROXIMATION

In the literature we often find a second-order differential equation of δ_{cv_c} or $\delta_{c\chi}$ derived in the small-scale limit [13, 35, 36]. By naively preserving only terms attached with k^2 except for δ_c term, Eq. (85) becomes

$$\ddot{\delta}_{cv_c} + 2H\dot{\delta}_{cv_c} \simeq -\frac{k^2}{a^2}\alpha_\chi. \quad (21)$$

From Eqs. (48) and (62) we have

$$\varphi_\chi + \alpha_\chi = -\frac{\delta F_\chi}{F}. \quad (22)$$

In the small-scale limit from Eq. (13), using Eqs. (17), (62) and (45), we have

$$\frac{k^2}{a^2} \left(\varphi_\chi + \frac{\delta F_\chi}{2F} \right) \simeq \frac{1}{2F}\delta\mu_{cv_c}. \quad (23)$$

We need one more relation. From the equation of motion in Eq. (64) and using Eqs. (65), (22), and $\delta F = F_{,\phi}\delta\phi + F_{,R}\delta R$ [here we consider $F_{,X} = 0$ case] we can derive

$$\begin{aligned} \delta R_\chi &\simeq -\frac{1}{F_{,\phi}} \left(f_{,\phi\phi} + f_{,X} \frac{k^2}{a^2} \right) \delta\phi_\chi \\ &\simeq \frac{-2\frac{k^2}{a^2} \left(\alpha_\chi + 2\frac{F_{,\phi}}{F} \delta\phi_\chi \right)}{1 + 4\frac{F_{,R}}{F} \frac{k^2}{a^2}}, \end{aligned} \quad (24)$$

where we have kept $f_{,\phi\phi}$ term compared with $f_{,X}k^2/a^2$ term, and order 1 compared with $(F_{,R}/F)k^2/a^2$ [see below Eq. (36)] as in [13, 35, 36]; we ignored $f_{,X\phi}\delta X$ term

in an expansion of $\delta f, \phi$ term in Eq. (64). From Eqs. (22)-(24), we have

$$\varphi_\chi \simeq - \frac{\left(1 + 2\frac{F_{,R}}{F}\frac{k^2}{a^2}\right) \left(f_{,\phi\phi} + f_{,X}\frac{k^2}{a^2}\right) - 2\frac{(F_{,\phi})^2}{F}\frac{k^2}{a^2}}{\left(1 + 4\frac{F_{,R}}{F}\frac{k^2}{a^2}\right) \left(f_{,\phi\phi} + f_{,X}\frac{k^2}{a^2}\right) - 4\frac{(F_{,\phi})^2}{F}\frac{k^2}{a^2}} \times \alpha_\chi \equiv -\bar{\eta}\alpha_\chi, \quad (25)$$

$$\frac{k^2}{a^2}\alpha_\chi \simeq -\frac{1}{2F} \frac{\left(1 + 4\frac{F_{,R}}{F}\frac{k^2}{a^2}\right) \left(f_{,\phi\phi} + f_{,X}\frac{k^2}{a^2}\right) - 4\frac{(F_{,\phi})^2}{F}\frac{k^2}{a^2}}{\left(1 + 3\frac{F_{,R}}{F}\frac{k^2}{a^2}\right) \left(f_{,\phi\phi} + f_{,X}\frac{k^2}{a^2}\right) - 3\frac{(F_{,\phi})^2}{F}\frac{k^2}{a^2}} \times \delta\mu_{cv_c}. \quad (26)$$

These can be compared with Eqs. (2) and (3). However, it is important to notice that $\delta\mu_{cv_c}$ differs from $\delta\mu_v$. By comparing Eqs. (13) and (23), and using Eqs. (22) and (25), we have

$$\delta\mu_v \simeq \frac{2\bar{\eta}}{1+\bar{\eta}} \frac{1}{F} \delta\mu_{cv_c}. \quad (27)$$

Using this, we have

$$\begin{aligned} \frac{k^2}{a^2}\alpha_\chi &\simeq -\frac{1}{1+\bar{\eta}} \frac{1}{F} \delta\mu_{cv_c} \\ &\simeq -\frac{1}{2\bar{\eta}} \delta\mu_v \equiv -\frac{1}{2}\bar{\mu}\delta\mu_v. \end{aligned} \quad (28)$$

Therefore, we have

$$\bar{\mu} \simeq \frac{1}{\bar{\eta}}, \quad (29)$$

which is an already presented result in an exact context in Eq. (14). The above relations are derived in the general context of $f(R, \phi, X)$ gravity which usually leads to higher order theory: see below Eq. (56).

From Eqs. (21) and (28) we have

$$\begin{aligned} \ddot{\delta}_{cv_c} + 2H\dot{\delta}_{cv_c} &\simeq \frac{1}{2}\bar{\mu}\delta\mu_v \simeq \frac{1}{2}\frac{2\bar{\mu}}{1+\bar{\mu}}\frac{1}{F}\delta\mu_{cv_c} \\ &\equiv 4\pi G_{\text{eff}}\delta\mu_{cv_c}, \end{aligned} \quad (30)$$

where the effective gravitational constant becomes

$$8\pi G_{\text{eff}} \simeq \frac{2\bar{\mu}}{1+\bar{\mu}} \frac{1}{F}. \quad (31)$$

For $f = R - 2X - 2V$ we have a correct Einstein's gravity result with $8\pi G_{\text{eff}} = 1$.

Now, we consider two cases: (i) $f = RF(\phi) + 2p(\phi, X)$ theory, and (ii) $f = f(R)$ theory.

(i) In the case of $f = RF(\phi) + 2p(\phi, X)$ theory, Eqs. (25) and (26) reduce to

$$\varphi_\chi \simeq - \frac{\left(f_{,X} - 2\frac{(F_{,\phi})^2}{F}\right) \frac{k^2}{a^2} + f_{,\phi\phi}}{\left(f_{,X} - 4\frac{(F_{,\phi})^2}{F}\right) \frac{k^2}{a^2} + f_{,\phi\phi}} \alpha_\chi, \quad (32)$$

$$\frac{k^2}{a^2}\alpha_\chi \simeq -\frac{1}{2F} \frac{\left(f_{,X} - 4\frac{(F_{,\phi})^2}{F}\right) \frac{k^2}{a^2} + f_{,\phi\phi}}{\left(f_{,X} - 3\frac{(F_{,\phi})^2}{F}\right) \frac{k^2}{a^2} + f_{,\phi\phi}} \delta\mu_{cv_c}. \quad (33)$$

(ii) In the case of $f = f(R)$ theory, however, the equation of motion in Eq. (64) is not available. Instead, from Eq. (65) we have

$$\delta R_\chi \simeq 2\frac{k^2}{a^2} (\alpha_\chi + 2\varphi_\chi) \simeq \frac{1}{F} \delta\mu_{c\chi} - 3\frac{k^2}{a^2} \frac{\delta F_\chi}{F}. \quad (34)$$

We take $\delta_{c\chi} \simeq \delta_{cv_c}$ in the small-scale limit. From Eqs. (34) and (22) we recover Eq. (23). Now, in the $f(R)$ theory we have $\delta F = F_{,R}\delta R$. Using this relation, from Eqs. (22) and (23) together with Eq. (34) we can derive

$$\varphi_\chi \simeq - \frac{1 + 2\frac{F_{,R}}{F}\frac{k^2}{a^2}}{1 + 4\frac{F_{,R}}{F}\frac{k^2}{a^2}} \alpha_\chi, \quad (35)$$

$$\frac{k^2}{a^2}\alpha_\chi \simeq -\frac{1}{2F} \frac{1 + 4\frac{F_{,R}}{F}\frac{k^2}{a^2}}{1 + 3\frac{F_{,R}}{F}\frac{k^2}{a^2}} \delta\mu_{cv_c}. \quad (36)$$

Although these results can be regarded as cases of Eqs. (25) and (26), we need a separate treatment because Eqs. (25) and (26) were derived from the equation of motion which is not available in $f(R)$ gravity. As in Eqs. (24) and (26) we kept $\delta R = \delta F/F_{,R}$ term even compared with $(k^2/a^2)\delta F/F$ term because $F_{,R}$ becomes zero in the Einstein gravity limit. Thus, it is important to notice that even in the small-scale limit we are already considering, we have to set $F_{,R}/F \ll a^2/k^2 \ll 1$ to properly have the Einstein's gravity limit.

Apparently, the derivation of Eq. (30) in the small-scale limit requires rather complicated steps, and the only way to check its validity is comparing with the exact result. In Fig. 1 we estimate the accuracy of the above asymptotic approximations using a case of $f(R)$ gravity. We consider a case with $f = R^{1+\epsilon} + qR^{-n}$. The $R^{1+\epsilon}$ term dominates in the early (radiation) era and allows a scaling evolution of μ_χ and $\delta\mu_\chi$ following the dominant fluid; first follows radiation and then matter. The qR^{-n} term can be tuned to cause accelerated evolution in recent era; we have studied this case in [44] in details. For the perturbations we solve equations in the CDM-comoving gauge presented in Eqs. (86)-(88) with additional presence of radiation component.

The Figure shows that in the small scale far inside the horizon the approximate equations in Eqs. (21) and (36) reproduce the evolution of δ_{cv_c} properly. That is, as the scale becomes smaller results from the small-scale approximation more properly reproduce ones from full calculation. The evolutions of G_{eff} , however, show that deviation of G_{eff} from Newtonian G is more significant as the scale becomes smaller, see Eq. (36). In practical applications it is always important to check the validity of such asymptotic form approximations compared with exact solutions based on the complete set of equations. The complete set of perturbation equations in several gauges in our generalized gravity theories is presented in the Appendix C.

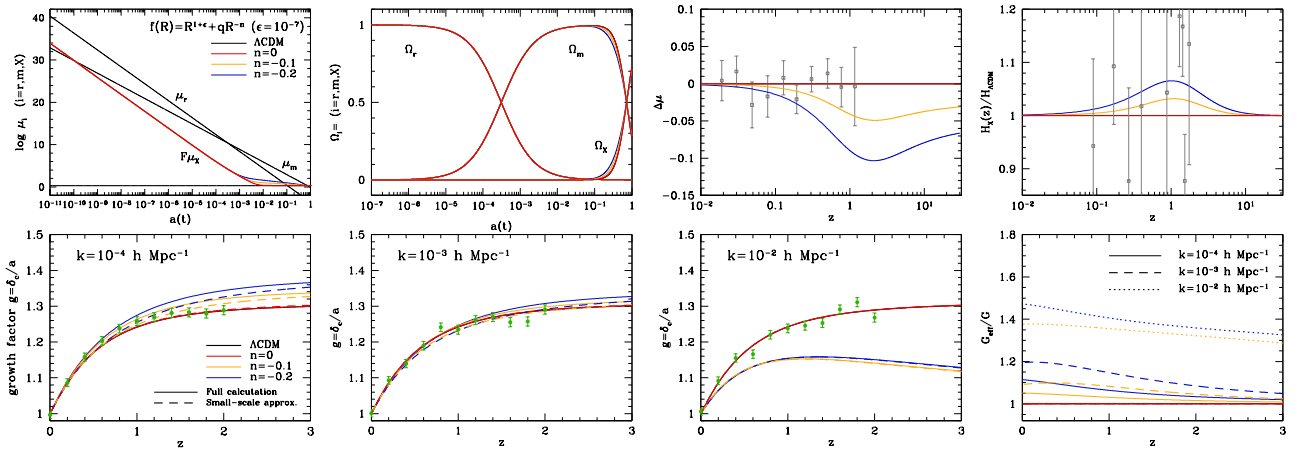


FIG. 1: Top panels from left to right: Evolution of density μ_i ($i = r, m, X$); recent behavior of density parameters due to $f(R)$ gravity; test under the type-Ia supernovae observation; evolution of the Hubble parameter. We use a functional form $f(R) = R^{1+\epsilon} + qR^{-n}$ with $\epsilon = 10^{-7}$ and $n = 0$ (red), -0.1 (yellow), -0.2 (blue curves). Note that we compare μ_r and μ_m with $F\mu_X$. Evolution of $F\mu_X$ shows early scaling and its recent domination as a dark energy. We use definitions, $\Omega_i \equiv \mu_i/(3FH^2)$ for radiation and matter and $\Omega_X \equiv \mu_X/(3H^2)$. The binned type-Ia supernovae data are based on the Union2 sample [45], and Hubble parameter data are from Ref. [46]. Bottom panels from left to right: Behaviors of the CDM density perturbation growth factor in three different scales; the last figure shows evolutions of G_{eff} . In the growth factor panels, results from the full calculation (including radiation; solid curves) and from the small scale approximation (dashed curves) are shown together. Green dots with 1% error bars are expectations from the future X-ray and weak lensing observations [47]. As the scale becomes smaller results from the small-scale approximation more properly reproduce ones from full calculation. The evolutions of G_{eff} , however, show that deviation of G_{eff} from Newtonian G is more significant as the scale becomes smaller.

IV. POTENTIAL COSMOLOGICAL SIGNATURES OF NON-EINSTEIN GRAVITY

Can we find evidence of non-Einstein gravity nature from the large scale cosmological observations? Before presenting a (in principle) possible way of using cosmological observations to distinguish potential non-Einstein' gravity nature, here, we would like to point out some potential loopholes in recent such arguments based on pseudo-Newtonian approaches motivated by approximate treatment of modified gravity theories.

One argument is based on the deviation of baryon (or CDM) density perturbation equation and Poisson's equation in modified gravity. In the small-scale limit from Eqs. (21), (30) and (31) we have

$$\ddot{\delta}_{cv_c} + 2H\dot{\delta}_{cv_c} \simeq -\frac{k^2}{a^2}\alpha_\chi, \quad (37)$$

$$\frac{k^2}{a^2}\alpha_\chi \simeq -4\pi G_{\text{eff}}\delta\mu_{cv_c}. \quad (38)$$

Under approximations used in previous section we have Eq. (31), and in the Einstein's gravity limit we have $8\pi G_{\text{eff}} = 1$. Here, it is important to remember that the closed form second-order equation in the considered modified gravity theories is valid only in the small-scale limit.

Although in Fig. 1 we have shown an example where the CDM (or baryon) growth rate is well approximated by the small-scale limit approximation in Eq. (30), that

is not necessarily the case for other cosmological observations like the CMB and the density power spectra, etc. It is important to notice that even in Einstein's gravity, in the presence of dynamic dark energy, in general, we cannot ignore the contribution of the dark energy perturbation in the right-hand-side of the Poisson's equation: see Eqs. (17) and (18). One such an example demonstrating the importance of coupling even in Einstein's gravity is presented in [48].

In [48] we have shown that even in the case of Einstein's gravity we have observationally significant deviations by ignoring the dark energy perturbations in the CMB and density power spectra, and in the baryon density perturbation growth rate. As we emphasized in this work, consideration of dynamical dark energy in general causes a coupling between the baryon (or CDM) density perturbation and the dark energy perturbation; and the dynamics of dark energy perturbation is described by its own additional equation of motion which is again coupled through gravity with other components. This is true even in the context of Einstein's gravity with a minimally coupled scalar field as a dynamical dark energy.

In [48] we have studied a case of dark energy model based on a minimally coupled scalar field with a double exponential potential as an example. We showed that proper consideration of the dark energy perturbations is important in comparing theories with present day cosmological observations. In the CMB temperature anisotropy power spectrum, the large-scale density power spectrum, and even in the perturbation growth rate we have shown

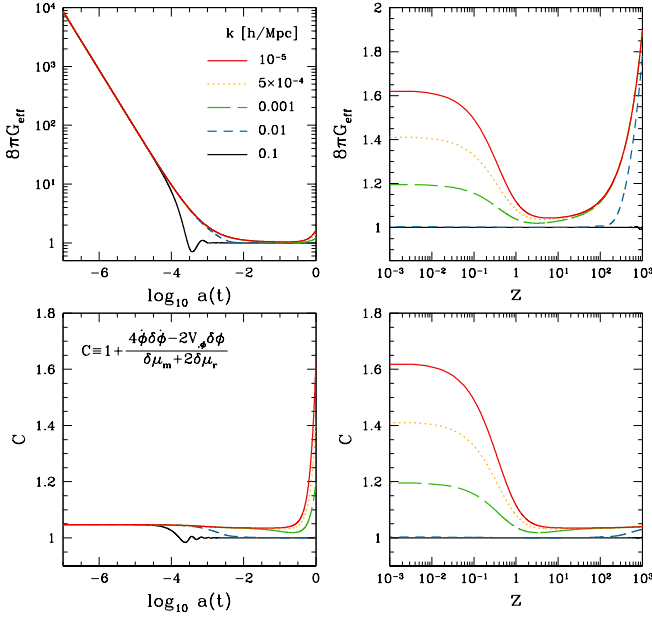


FIG. 2: Top panels: Evolution of $8\pi G_{\text{eff}}$ as a function of scale factor $a(t)$ and the redshift z for several different scales. As the scale becomes smaller $8\pi G_{\text{eff}}$ approaches unity. We consider a double exponential potential $V(\phi) = V_1 e^{-\lambda_1 \phi} + V_2 e^{-\lambda_2 \phi}$ where ϕ is the scalar field and $\lambda_2 = 1.0$; for other parameters, see [48]. Bottom panels: Evolution of the ratio of the scalar field compared with the ordinary fluid (matter plus radiation) using the same model as in the top panel. Deviations of the ratio from unity in the early radiation and matter dominated eras are due to the scaling nature of scalar field we considered.

that it is critically important to take into account of dark energy perturbations properly. This result can be regarded as an evidence that the coupling between the baryon (or CDM) density perturbation and the dynamical dark energy perturbation is in general important even in Einstein's gravity. In the presence of a scalar field dark energy, in the CCG, from Eqs. (49), (54), (55) and (62) we have

$$\begin{aligned} \ddot{\delta}_c + 2H\dot{\delta}_c &= \frac{1}{2}(\delta\mu + 3\delta p) \\ &= \frac{1}{2}(\delta\mu_c + 2\delta\mu_r + 4\dot{\phi}\delta\dot{\phi} - 2V_{,\phi}\delta\phi) \\ &\equiv 4\pi G_{\text{eff}}\delta\mu_c. \end{aligned} \quad (39)$$

In Fig. 2 we present evolution of $8\pi G_{\text{eff}}$ and the effect of the scalar field contribution using a double exponential potential model of a minimally coupled scalar field in Einstein's gravity [48]. Although becomes negligible as we go to the small scale, we have $8\pi G_{\text{eff}} \neq 1$ even for a minimally coupled scalar field as the dark energy in Einstein's gravity.

The other argument is based on Eq. (2) so that $\bar{\eta} \neq 1$ could be regarded as a strong signature of the non-Einstein gravity nature. The main point used in this

argument is that the gravitational weak lensing effect can measure the *sum* of Newtonian potential ($-\alpha_\chi$) and the post-Newtonian potential (φ_χ). From Eq. (48) we have

$$\alpha_\chi + \varphi_\chi = -\Pi, \quad (40)$$

Although we could have significant amount of anisotropic stress Π even in Einstein's gravity, that could be indeed regarded as highly peculiar situation almost comparable to rather considering the non-Einstein gravity. As we have $\varphi_\chi = -\alpha_\chi$ in Einstein's gravity, we can accept that it is reasonable to regard $\Pi = (\bar{\eta} - 1)\alpha_\chi \neq 0$, thus $\bar{\eta} \neq 1$ as a strong signature of non-Einstein's gravity. Now, as $\alpha_\chi - \varphi_\chi$ can be measured from the weak lensing observation, the remaining problem is whether we can determine α_χ or φ_χ individually in Einstein's gravity.

Here, it is interesting to point out that the evolution of baryonic velocity perturbation $v_{b\chi}$, in principle, can determine the Newtonian potential α_χ [32, 34]. Independently of the gravity theories, from Eq. (55) or (67) for the baryon component, we have

$$\dot{v}_{b\chi} + H v_{b\chi} = \frac{k}{a} \alpha_\chi. \quad (41)$$

Similar equation for the CDM component is presented in Eq. (11). Thus, apparently, the velocity perturbation depends only on the Newtonian gravitational potential α_χ . Since this equation is derived from the momentum conservation of the baryon component, unless there exist exotic interaction with other components, this equation is generally valid independently of the nature of the gravity theory based on Riemannian geometry.

Therefore, in Einstein's gravity theory we have

$$\alpha_\chi - \varphi_\chi = 2\alpha_\chi. \quad (42)$$

In this expression the left-hand-side and the right-hand-side can be determined from weak lensing observation and peculiar velocity field observation, respectively. By combining these two informations we can, in principle, tell whether the post-Newtonian gravitational potential (φ_χ) has the same amplitude as the Newtonian potential ($-\alpha_\chi$). If the two amplitudes differ from each other, i.e. if Eq. (42) is violated, assuming validity of linear theory, that can be regarded as an important signature of non-Einstein's gravity nature revealed in the cosmological situation. For recent study in this direction, see [34].

V. DISCUSSION

We have considered feasibility of modifying Newton's gravity in cosmology from the perspective of generalized gravity theories. As a concrete example we examined $f(R, \phi, X)$ gravity which includes $f(R)$ gravity and many other gravity theories known in the literature as cases. For the baryon or CDM component in the context of the generalized gravity theory, the perturbation equations

are inevitably coupled with the presence of the generalized gravity theory sector, thus in general ending up with at least fourth-order differential equation; situation is the same even in the minimally coupled scalar field as a dark energy in Einstein's gravity. The closed form of CDM density perturbation equation proposed in the literature is only possible in the small-scale limit in the context of generalized relativistic gravity theories. Even in such a small-scale limit we can easily introduce cases where proper treatment of generalized gravity is important to compare with the CMB and density power spectra.

It is always important to analyze properly (i.e., without approximation) the roles of dynamic dark energy perturbation in Einstein's gravity before one concludes against the canonical gravity theory which has been observationally successful in all weak gravity tests. We also have addressed potential loopholes in arguing non-Einstein gravity nature based on cosmologically modified pseudo-Newtonian gravity theories. We point out that future precise observations of the weak lensing together with the baryon velocity perturbation can potentially test non-Einstein gravity nature based on cosmological observations.

Appendix

Appendix A: Perturbations in relativistic gravity

In the Appendix A and B we consider the presence of general K . Background evolution is described by

$$H^2 = \frac{1}{3}\mu - \frac{K}{a^2}, \quad \dot{\mu} = -3H(\mu + p). \quad (43)$$

In the multiple component case, we have $\mu = \sum_j \mu_j$ and $p = \sum_j p_j$, together with

$$\dot{\mu}_i = -3H(\mu_i + p_i) + I_{0i}, \quad (44)$$

where $\sum_j I_{0j} \equiv 0$. Based on our metric and energy-momentum tensor conventions in Eqs. (4) and (5), the scalar-type perturbations are described by [49]

$$\kappa \equiv -3\dot{\varphi} + 3H\alpha + \frac{k^2}{a^2}\chi, \quad (45)$$

$$\frac{1}{2}\delta\mu + H\kappa = \frac{k^2 - 3K}{a^2}\varphi, \quad (46)$$

$$\kappa - \frac{k^2 - 3K}{a^2}\chi = \frac{3}{2}(\mu + p)\frac{a}{k}v, \quad (47)$$

$$\dot{\chi} + H\chi - \varphi - \alpha = \Pi, \quad (48)$$

$$\dot{\kappa} + 2H\kappa + \left(3\dot{H} - \frac{k^2}{a^2}\right)\alpha = \frac{1}{2}(\delta\mu + 3\delta p), \quad (49)$$

$$\delta\dot{\mu} + 3H(\delta\mu + \delta p) = (\mu + p)\left(\kappa - 3H\alpha - \frac{k}{a}v\right), \quad (50)$$

$$\frac{[a^4(\mu + p)v]'}{a^4(\mu + p)} = \frac{k}{a}\alpha + \frac{k}{a(\mu + p)}\left(\delta p - \frac{2}{3}\frac{k^2 - 3K}{a^2}\Pi\right), \quad (51)$$

where $\chi \equiv a\beta + a^2\dot{\gamma}$ is a spatially gauge-invariant combination. We note that all our perturbation equations are spatially gauge invariant. Although derived in Einstein's gravity these equations are valid even in the context of generalized gravity theories including the $f(R, \phi, X)$ gravity presented in the Appendix B. The generalized gravity theory we consider can be written in the Einstein's gravity form

$$G_{ab} = T_{ab}, \quad (52)$$

where T_{ab} is reinterpreted as the effective energy-momentum tensor [50]. In this way, as long as the generalized gravity can be expressed as in Eq. (52), the above basic set of perturbation equations is valid in those gravity theories as well.

In the presence of multiple component fluids (including dust, dark matter, radiation, fields, etc.), we have

$$\begin{aligned} \delta\mu &= \sum_j \delta\mu_j, & \delta p &= \sum_j \delta p_j, \\ (\mu + p)v &= \sum_j (\mu_j + p_j)v_j, & \Pi &= \sum_j \Pi_j. \end{aligned} \quad (53)$$

In the case of generalized gravity theory, see Eqs. (61) and (63). The individual component follows its own equation of motion

$$\delta\dot{\mu}_i + 3H(\delta\mu_i + \delta p_i) = (\mu_i + p_i) \left(\kappa - 3H\alpha - \frac{k}{a}v_i \right) - \frac{1}{a}\delta I_{0i}, \quad (54)$$

$$\frac{[a^4(\mu_i + p_i)v_i]}{a^4(\mu_i + p_i)} = \frac{k}{a}\alpha + \frac{k}{a(\mu_i + p_i)} \left(\delta p_i - \frac{2}{3}\frac{k^2 - 3K}{a^2}\Pi_i - \delta I_i \right), \quad (55)$$

where $\sum_j \delta I_{0j} \equiv 0 \equiv \sum_j \delta I_j$.

Appendix B: Perturbations in $f(R, \phi, X)$ gravity

We consider $f(R, \phi, X)$ gravity considered in [50]. The action is

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2}f(R, \phi, X) + L_m \right], \quad (56)$$

where $X \equiv \frac{1}{2}\dot{\phi}^c\dot{\phi}_{,c}$. The minimally coupled scalar field is a case with $f = R - 2X - 2V(\phi)$. Even in the absence of additional matter component the $f(R, \phi, X)$ gravity, in general, leads to a fourth-order differential equation for the scalar-type perturbations. In order to have second-order differential equation with nontrivial $F \equiv \partial f / \partial R$ we should have either (i) $f = RF(\phi, X) + 2p(\phi, X)$ or (ii) $f = f(R)$, see [50].

The gravitational field equation following from the above action can be arranged in the form of Einstein's gravity in which the new contributions are interpreted as effective energy-momentum tensor; the effective fluid quantities are introduced in Eq. (5). Using this strategy, the background and perturbation equations in Einstein's gravity remain valid with the fluid quantities replaced by the effective ones [51].

Background evolution is still described by Eq. (43) with the fluid quantities reinterpreted as the following

effective ones

$$\begin{aligned} \mu &= \frac{1}{F}\mu_m + \mu_X, \quad p = \frac{1}{F}p_m + p_X, \\ \mu_X &\equiv \frac{1}{F} \left(f_{,X}X + \frac{FR - f}{2} - 3H\dot{F} \right), \\ p_X &\equiv \frac{1}{F} \left(-\frac{FR - f}{2} + \ddot{F} + 2H\dot{F} \right), \end{aligned} \quad (57)$$

where $X \equiv -\frac{1}{2}\dot{\phi}^2$. The equations of motion is presented in Eq. (80) of [50]

$$\frac{1}{a^3} \left(a^3 \dot{\phi}_{,X} \right)' + f_{,\phi} = 0. \quad (58)$$

We also have

$$R = 6 \left(2H^2 + \dot{H} + \frac{K}{a^2} \right) = \mu - 3p. \quad (59)$$

The ordinary matter part follows its own equation of motion $\dot{\mu}_m = -3H(\mu_m + p_m)$. Thus, we have

$$\dot{\mu}_X + 3H(\mu_X + p_X) = \frac{\dot{F}}{F^2}\mu_m. \quad (60)$$

In our generalized gravity context Eqs. (45)-(51) remain valid with the fluid quantities reinterpreted as the following effective ones

$$\begin{aligned} \delta\mu &= \frac{1}{F}\delta\mu_m + \delta\mu_X - \mu_m \frac{\delta F}{F^2}, \quad \delta p = \frac{1}{F}\delta p_m + \delta p_X - p_m \frac{\delta F}{F^2}, \\ (\mu + p)v &= \frac{1}{F}(\mu_m + p_m)v_m + (\mu_X + p_X)v_X, \quad \Pi = \frac{1}{F}\Pi_m + \Pi_X, \end{aligned} \quad (61)$$

with

$$\begin{aligned} \delta\mu_X &= \frac{1}{F} \left[\frac{1}{2}(f_{,X}\delta X - f_{,\phi}\delta\phi) + \delta f_{,X}X - 3H\delta\dot{F} + \left(\frac{1}{2}f - f_{,X}X + 3H\dot{F} - F\frac{k^2}{a^2} \right) \frac{\delta F}{F} + \dot{F}\kappa + 3H\dot{F}\alpha \right], \\ \delta p_X &= \frac{1}{F} \left[\frac{1}{2}(f_{,X}\delta X + f_{,\phi}\delta\phi) + \delta\ddot{F} + 2H\delta\dot{F} - \left(\frac{1}{2}f + \ddot{F} + 2H\dot{F} - F\frac{2}{3}\frac{k^2}{a^2} \right) \frac{\delta F}{F} - \frac{2}{3}\dot{F}\kappa - \dot{F}\alpha - 2(\ddot{F} + H\dot{F})\alpha \right], \\ (\mu_X + p_X)v_X &= -\frac{k}{aF} \left(\frac{1}{2}f_{,X}\dot{\phi}\delta\phi - \delta\dot{F} + H\delta F + \dot{F}\alpha \right), \\ \Pi_X &= \frac{1}{F}(\delta F - \dot{F}\chi), \end{aligned} \quad (62)$$

and $\delta X = -\dot{\phi}\delta\phi + \dot{\phi}^2\alpha$. If we have multiple fluid components, we have

$$\begin{aligned}\mu_m &= \sum_j \mu_j, & p_m &= \sum_j p_j, \\ \delta\mu_m &= \sum_j \delta\mu_j, & \delta p_m &= \sum_j \delta p_j, & (\mu_m + p_m)v_m &= \sum_j (\mu_j + p_j)v_j, & \Pi_m &= \sum_j \Pi_j,\end{aligned}\quad (63)$$

where the individual component satisfies Eq. (44) for the background and Eqs. (54) and (55) for the perturbations. The equation of motion is presented in Eq. (81) of [50]

$$f_{,X} \left[\delta\ddot{\phi} + \left(3H + \frac{(f_{,X})}{f_{,X}} \right) \delta\dot{\phi} + \frac{k^2}{a^2} \delta\phi + \dot{\phi} \left(3\dot{\phi} - \dot{\alpha} - \frac{k^2}{a^2} \chi \right) \right] + 2f_{,\phi}\alpha + \frac{1}{a^3} \left(a^3 \dot{\phi} \delta f_{,X} \right)' + \delta f_{,\phi} = 0. \quad (64)$$

We also have

$$\delta R = 2 \left[-\dot{\kappa} - 4H\kappa + \left(\frac{k^2}{a^2} - 3\dot{H} \right) \alpha + 2 \frac{k^2 - 3K}{a^2} \varphi \right] = \delta\mu - 3\delta p. \quad (65)$$

If the ordinary matter part follows its own equation of motion

$$\delta\dot{\mu}_m + 3H(\delta\mu_m + \delta p_m) = (\mu_m + p_m) \left(\kappa - 3H\alpha - \frac{k}{a} v_m \right), \quad (66)$$

$$\frac{[a^4(\mu_m + p_m)v_m]'}{a^4(\mu_m + p_m)} = \frac{k}{a}\alpha + \frac{k}{a(\mu_m + p_m)} \left(\delta p_m - \frac{2}{3} \frac{k^2 - 3K}{a^2} \Pi_m \right), \quad (67)$$

which follow from Eqs. (54) and (55), from Eqs. (50) and (51) we have

$$\delta\dot{\mu}_X + 3H(\delta\mu_X + \delta p_X) - (\mu_X + p_X) \left(\kappa - 3H\alpha - \frac{k}{a} v_X \right) = \frac{\dot{F}}{F^2} \delta\mu_m + \mu_m \frac{1}{F^2} \left(\delta\dot{F} - 2\frac{\dot{F}}{F} \delta F \right), \quad (68)$$

$$\frac{1}{a^4} [a^4(\mu_X + p_X)v_X]' - \frac{k}{a} \left[(\mu_X + p_X)\alpha + \delta p_X - \frac{2}{3} \frac{k^2 - 3K}{a^2} \Pi_X \right] = \frac{\dot{F}}{F^2} (\mu_m + p_m)v_m - \frac{1}{F^2} \frac{k}{a} p_m \delta F. \quad (69)$$

Notice that in the single component case without m -component, Eqs. (68) and (69) are the same as Eqs. (66) and (67) with the sub-indices m replaced by X .

Appendix C: CDM perturbation in $f(R, \phi, X)$ gravity

Here we set $K = 0$. For simplicity, we consider (i) $f = RF(\phi) + 2p(\phi, X)$ and (ii) $f = f(R)$. In the following, we present complete sets of equations of CDM in the above generalized gravity theories, in three different gauge conditions. These are the zero-shear gauge, the uniform-field (or F) gauge, and the CDM-comoving gauge.

Appendix C.1: Zero-shear gauge

The zero-shear gauge takes $\chi \equiv 0$ as the temporal gauge condition. Equations (10) and (11) are valid for differential equations for $\delta_{c\chi}$ and $v_{c\chi}$. Using Eq. (12) α_χ can be expressed in terms of φ_χ and δF_χ . Thus, two additional first-order differential equations for φ_χ and δF_χ will complete the perturbation equations. One equation follows from Eqs. (45) and (47)

$$\dot{\phi} + \left(H + \frac{\dot{F}}{2F} \right) \varphi = \frac{1}{2F} \left[\frac{1}{2} f_{,X} \dot{\phi} \delta\phi - \delta\dot{F} - \frac{1}{2} \left(H + \frac{\dot{F}}{F} \right) \frac{\delta F}{F} - \mu_c \frac{a}{k} v_c \right]. \quad (70)$$

The other one can be derived from Eq. (13)

$$\begin{aligned}
& \frac{1}{2} (f_{,X} + 2f_{,XX}X) \dot{\phi} \delta\dot{\phi} - \frac{3\dot{F}}{2F} \delta\dot{F} + \frac{1}{2} \left[f_{,\phi} - 2f_{,\phi X}X + 3 \left(H + \frac{\dot{F}}{2F} \right) f_{,X} \dot{\phi} \right] \delta\phi \\
& + \left[3H^2 + \frac{\mu_c}{F} - \frac{3\dot{F}}{2F} H - \frac{1}{F} \left(\frac{3\dot{F}^2}{2F} + \frac{1}{2} f + 2f_{,XX}X^2 \right) + \frac{k^2}{a^2} \right] \delta F \\
& = \left(\frac{3\dot{F}^2}{2F} + f_{,X}X + 2f_{,XX}X^2 - 2F \frac{k^2}{a^2} \right) \varphi + \delta\mu_c + 3 \left(H + \frac{\dot{F}}{2F} \right) \frac{a}{k} \mu_c v_c,
\end{aligned} \tag{71}$$

where we have $\delta F = F_{,\phi} \delta\phi$ for (i), and $\delta\phi = 0 = X$ for (ii). From these we have the two equations for $\dot{\varphi}_\chi$ and $\delta\dot{F}_\chi$. And these two equations are the additional equations we need to solve together with Eqs. (10) and (11).

Appendix C.2: Uniform-field gauge

We take the uniform-field gauge for (i), and the uniform- F gauge for (ii); in both cases we set $\delta\phi = 0 = \delta F$ as the temporal gauge condition, and we call the gauge condition as UFG. We have $\delta X = -2X\alpha$ and $\delta f_{,X} = -2f_{,XX}X\alpha$. From Eqs. (45)-(47), Eqs. (45), (47), and (48), respectively, we have

$$\begin{aligned}
\dot{\Phi} &= \frac{\dot{F} + 2HF}{\frac{3\dot{F}^2}{2F} + f_{,X}X + 2f_{,XX}X^2} \left[-\frac{k^2}{a^2} \Psi + \frac{1}{2F} \delta\mu_c \right. \\
&\quad \left. + \left(H + \frac{\dot{F}}{2F} \right) \frac{3a}{2F} \mu_c v_c \right] - \frac{1}{2F} \frac{a}{k} \mu_c v_c,
\end{aligned} \tag{72}$$

$$\begin{aligned}
\frac{H + \frac{\dot{F}}{2F}}{aF} \left(\frac{aF}{H + \frac{\dot{F}}{2F}} \Psi \right) &= \frac{\frac{3\dot{F}^2}{2F} + f_{,X}X}{\dot{F} + 2HF} \Phi \\
&+ \frac{\mu_c}{\dot{F} + 2HF} \Psi - \frac{1}{2F} \mu_c \left(\frac{a}{k} v_c - \chi \right),
\end{aligned} \tag{73}$$

where

$$\Phi \equiv \varphi_{\delta F}, \quad \Psi \equiv \varphi_\chi + \frac{\delta F_\chi}{2F}. \tag{74}$$

In the context of $f(R)$ gravity we simply set $X \equiv 0$. In the absence of the CDM component, these equations are presented in Eqs. (85) and (86) of [50]. For the CDM part, from Eq. (54) and Eq. (55), respectively, we have

$$\dot{\delta}_c = -\frac{k}{a} v_c + \kappa - 3H\alpha, \tag{75}$$

$$\dot{v}_c + H v_c = \frac{k}{a} \alpha, \tag{76}$$

where from Eqs. (45)-(47) we have

$$\alpha = \frac{1}{H + \frac{\dot{F}}{2F}} \left(\dot{\Phi} + \frac{1}{2F} \frac{a}{k} \mu_c v_c \right), \tag{77}$$

$$\begin{aligned}
\kappa &= \frac{1}{H + \frac{\dot{F}}{2F}} \left[\frac{k^2}{a^2} \Phi - \frac{1}{2F} \delta\mu_c \right. \\
&\quad \left. - \frac{1}{2F} \left(3H\dot{F} - f_{,X}X - 2f_{,XX}X^2 \right) \alpha \right],
\end{aligned} \tag{78}$$

$$\chi = \frac{a^2}{k^2} \left(-\frac{3}{2F} \frac{a}{k} \mu_c v_c + \kappa + \frac{3}{2F} \dot{F} \alpha \right). \tag{79}$$

These equations give a set of differential equations for Φ , Ψ , δ_c and v_c ; these can be combined to give a fourth-order differential equation. Φ and Ψ are gauge-invariant combinations, and all the other perturbation variables are evaluated in the UFG; δ_c and v_c evaluated in the UFG are equivalent to gauge-invariant combinations $\delta_{c\delta F} \equiv \delta_c + (3H\dot{F})\delta F$ and $v_{c\delta F} \equiv v_c - (1/a\dot{F})\delta F$, respectively. The CDM density perturbation in the CDM-comoving gauge can be constructed by

$$\delta_{cv_c} \equiv \delta_c + 3H \frac{a}{k} v_c, \tag{80}$$

where the right-hand-side can be evaluated in the UFG.

Appendix C.3: CDM-comoving gauge

The CDM-comoving gauge (CCG) takes $v_c \equiv 0$ as the temporal gauge condition. From Eq. (55) we have $\alpha = 0$. Equations (54) and (47), Eq. (48), and Eqs. (45) and Eq. (47), respectively, give

$$\dot{\delta}_c = \frac{k^2}{a^2} \chi - \frac{3}{2F} \left(\frac{1}{2} f_{,X} \dot{\phi} \delta\phi - \delta\dot{F} + H\delta F \right), \tag{81}$$

$$\dot{\chi} + \left(H + \frac{\dot{F}}{F} \right) \chi = \varphi + \frac{\delta F}{F}, \tag{82}$$

$$\dot{\varphi} = \frac{1}{F} \left(\frac{1}{2} f_{,X} \dot{\phi} \delta\phi - \delta\dot{F} + H\delta F \right). \tag{83}$$

In order to complete we need a first-order differential equation for δF . From Eqs. (46) and (47) we have

$$\begin{aligned} & \frac{1}{2} (f_{,X} + 2f_{,XX}X) \dot{\phi} \delta \dot{\phi} - \frac{3\dot{F}}{2F} \delta \dot{F} + \frac{1}{2} \left[f_{,\phi} - 2f_{,\phi X}X + 3 \left(H + \frac{\dot{F}}{2F} \right) f_{,X} \dot{\phi} \right] \delta \phi \\ & + \left[3H^2 + \frac{\mu_c}{F} - \frac{3\dot{F}}{2F} H - \frac{1}{F} \left(\frac{1}{2} f - f_{,X}X \right) + \frac{k^2}{a^2} \right] \delta F = -2F \frac{k^2}{a^2} \varphi + \delta \mu_c + \left(\dot{F} + 2HF \right) \frac{k^2}{a^2} \chi, \end{aligned} \quad (84)$$

where we have $\delta F = F_{,\phi} \delta \phi$ for (i), and $\delta \phi = 0 = X$ for (ii). Equations (81)-(84) provide a complete set of equations for δ_c , χ , φ and δF .

Equations (81)-(83) can be combined to give

$$\frac{1}{a^2} \left[a^2 (\delta_c + 3\varphi) \right]' = -\frac{k^2}{a^2} \alpha_\chi, \quad (85)$$

where $\delta_c + 3\varphi \equiv \delta_{c\varphi} \equiv 3\varphi_{\delta_c}$ is a gauge-invariant combination, thus can be evaluated in any gauge condition;

$\delta_{c\varphi}$ is δ_c in the uniform-curvature gauge ($\varphi \equiv 0$) and φ_{δ_c} is the spatial curvature perturbation φ in the uniform-CDM-density gauge ($\delta_c \equiv 0$).

In the case of $f(R)$ gravity we suggest the following alternative form. From $\delta R = \delta \mu - 3\delta p$, Eq. (49), and Eq. (54) we have

$$\delta \ddot{F} + 3H \delta \dot{F} + \left(-\frac{1}{3}R + \frac{k^2}{a^2} \right) \delta F + \frac{1}{3} F \delta R = \dot{F} \kappa + \frac{1}{3} (\delta \mu_m - 3\delta p_m), \quad (86)$$

$$\dot{\kappa} + \left(2H - \frac{\dot{F}}{F} \right) \kappa = -3H \frac{\delta \dot{F}}{F} + \left(\frac{1}{2} f + 3H \dot{F} - \mu_m - F \frac{k^2}{a^2} \right) \frac{\delta F}{F^2} - \frac{1}{2} \delta R + \frac{1}{F} \delta \mu_m, \quad (87)$$

$$\dot{\delta}_c = \kappa. \quad (88)$$

For a CDM component we have $\mu_m = \mu_c$, $\delta \mu_m = \mu_c \delta_c$, and $\delta p_m = 0$.

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